CHANGES OF COORDINATE

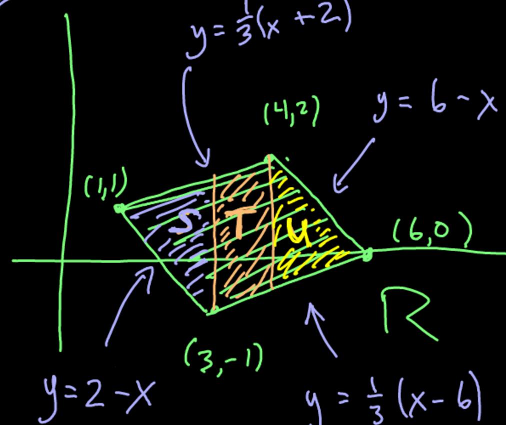
IDEA: Generalize the substitution method from Calculus I.

Ex: Compute $\iint_R (x+y)\,dA$ where $R \subseteq \mathbb{R}^2$ is the parallelogram with vertices $(1,1)$, $(3,-1)$, $(6,0)$, and $(4,2)$

Sol: $R = S \cup T \cup U$

NB: $m = \frac{\Delta y}{\Delta x}$

PICTURE



$S = \{(x,y) : 1 \le x \le 3,\ 2-x \le y \le \frac{1}{3}(x+2)\}$

$T = \{(x,y) : 3 \le x \le 4,\ \frac{1}{3}(x-6) \le y \le \frac{1}{3}(x+2)\}$

$U = \{(x,y) : 4 \le x \le 6,\ \frac{1}{3}(x-6) \le y \le 6-x\}$

$$\iint_R (x+y)\,dA = \iint_S (x+y)\,dA + \iint_T (x+y)\,dA + \iint_U (x+y)\,dA$$

$S = \{(x,y) : 1 \le x \le 3,\ 2-x \le y \le \frac{1}{3}(x+2)\}$

$$\iint_S (x+y)\,dA = \int_{x=1}^{3} \int_{y=2-x}^{\frac{1}{3}(x+2)} (x+y)\,dy\,dx$$

$$= \int_{x=1}^{3} \left[xy + \frac{1}{2}y^2 \right]_{y=2-x}^{\frac{1}{3}(x+2)} dx$$

$$= \int_{x=1}^{3} \left(\left(x \cdot \frac{1}{3}(x+2) + \frac{1}{2}\left(\frac{1}{3}(x+2)\right)^2 \right) - \left(x(2-x) + \frac{1}{2}(2-x)^2 \right) \right) dx$$

$$= \int_{x=1}^{3} \left(\frac{6}{18}(x^2+2x) + \frac{1}{18}(x^2+4x+4) - \left(2x - x^2 + \frac{1}{2}(4-4x+x^2)\right) \right) dx$$

$$= \int_{x=1}^{3} \left(\frac{1}{18}\left(6x^2+12x + x^2+4x+4\right) + \frac{18}{18}\left(-2x+x^2\right) + \frac{9}{18}\left(-4+4x-x^2\right) \right) dx$$

$$= \frac{1}{18}\int_{x=1}^{3}\left(7x^2 + 16x + 4 \cancel{-36x} + 18x^2 - 36 \cancel{+36x} - 9x^2\right) dx$$

$$= \frac{1}{18}\int_{x=1}^{3}\left(16x^2 + 16x - 32\right) dx$$

$$= \frac{16}{18}\int_{x=1}^{3}\left(x^2 + x - 2\right) dx = \frac{8}{9}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{x=1}^{3}$$

$$= \frac{8}{9}\left(\left(\frac{27}{3} + \frac{9}{2} - 6\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)\right)$$

$$= \frac{8}{9}\left(3 + \frac{9}{2} + 2 - \frac{1}{3} - \frac{1}{2}\right) = \frac{8}{9}\left(5 + 4 - \frac{1}{3}\right)$$

$$= \frac{8}{9}\left(\frac{27}{3} - \frac{1}{3}\right) = \frac{8\cdot 26}{27} = \underline{\frac{208}{27}}$$

$$T = \left\{(x,y) : 3 \le x \le 4,\ \frac{1}{3}(x-6) \le y \le \frac{1}{3}(x+2)\right\}$$

$$\iint_T (x+y)\, dA = \int_{x=3}^{4}\int_{y=\frac{1}{3}(x-6)}^{\frac{1}{3}(x+2)} (x+y)\, dy\, dx$$

$$= \int_{x=3}^{4}\left[xy + \frac{1}{2}y^2\right]_{y=\frac{1}{3}(x-6)}^{\frac{1}{3}(x+2)} dx$$

$$= \int_{x=3}^{4}\left(\left(x\cdot\frac{1}{3}(x+2) + \frac{1}{2}\left(\frac{1}{3}(x+2)\right)^2\right) - \left(x\cdot\frac{1}{3}(x-6) + \frac{1}{2}\left(\frac{1}{3}(x-6)\right)^2\right)\right) dx$$

$$= \int_{x=3}^{4}\left(\frac{6}{18}(x^2+2x) + \frac{1}{18}(x^2+4x+4) - \frac{6}{18}(x^2-6x) - \frac{1}{18}(x^2-12x+36)\right) dx$$

$$= \frac{1}{18}\int_{x=3}^{4}\left(\cancel{6x^2} + 12x \cancel{+x^2} + 4x + 4 \cancel{-6x^2} + 36x \cancel{-x^2} + 12x - 36\right) dx$$

$$= \frac{1}{18}\int_{x=3}^{4}\left(64x - 32\right) dx = \frac{32}{18}\int_{x=3}^{4}\left(2x - 1\right) dx$$

$$= \frac{16}{9}\left[x^2 - x\right]_{x=3}^{4} = \frac{16}{9}\left((16-4) - (9-3)\right) = \frac{16}{9}(12-6) = \frac{16}{9}\cdot 6 = \underline{\frac{32}{3}}$$

$U = \{(x,y) : 4 \le x \le 6,\ \frac{1}{3}(x-6) \le y \le 6-x\}$

$$\iint_U (x+y)\,dA = \int_{x=4}^{6} \int_{y=\frac{1}{3}(x-6)}^{6-x} (x+y)\,dy\,dx$$

$$= \int_{x=4}^{6} \left[xy + \frac{1}{2}y^2 \right]_{y=\frac{1}{3}(x-6)}^{6-x} dx$$

$$= \int_{x=4}^{6} \left(\left(x(6-x) + \frac{1}{2}(6-x)^2\right) - \left(x \cdot \frac{1}{3}(x-6) + \frac{1}{2}\left(\frac{1}{3}(x-6)\right)^2\right) \right) dx$$

$$= \int_{x=4}^{6} \left(x(6-x) + \frac{1}{3}x(6-x) + \frac{1}{2}(6-x)^2 - \frac{1}{18}(6-x)^2 \right) dx$$

$$= \int_{x=4}^{6} \left(\frac{24}{18}(6x - x^2) + \frac{8}{18}(x^2 - 12x + 36) \right) dx$$

$$= \frac{8}{18} \int_{x=4}^{6} (18x - 3x^2 + x^2 - 12x + 36)\,dx$$

$$= \frac{4}{9} \int_{x=4}^{6} (-2x^2 + 6x + 36)\,dx = \frac{4}{9}\left[-\frac{2}{3}x^3 + 3x^2 + 36x \right]_{x=4}^{6}$$

$$= \frac{4}{9}\left(\left(-\frac{2 \cdot 216}{3} + 3 \cdot 36 + 216 \right) - \left(-\frac{2}{3} \cdot 64 + 3 \cdot 16 + 36 \cdot 4 \right) \right)$$

$$= \frac{4}{9}\left(216\left(1 - \frac{2}{3}\right) + 108 + \frac{128}{3} - 48 - 144 \right)$$

$$= \frac{4}{9}\left(\frac{3 \cdot 72}{3} + 60 + \frac{128}{3} - 144 \right) = \frac{4}{9}\left(\frac{128}{3} + 60 - 72 \right) = \frac{4}{9}\left(\frac{128 - 12 \cdot 3}{3} \right)$$

$$= \frac{4}{9} \cdot \frac{92}{3} = \frac{368}{27}$$

$$\therefore \iint_R (x+y)\,dA = \iint_S (x+y)\,dA + \iint_T (x+y)\,dA + \iint_U (x+y)\,dA$$

$$= \frac{208}{27} + \frac{32}{3} + \frac{368}{27} = \frac{208 + 288 + 368}{27}$$

$$= \frac{496 + 368}{27} = \frac{864}{27} = \frac{9 \cdot 96}{9 \cdot 3} = \frac{3 \cdot 32}{3} = 32 \quad \boxed{//}$$

Even though the above example was somewhat nasty, there's no reason it should have been: the parallelogram is a nice shape, and the integrand is simple. We'll use this to make the integral easier...

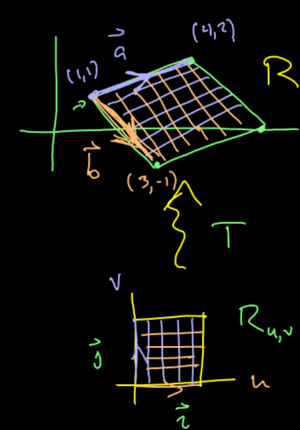
Key Idea: Make the region into something easier via **Coordinate transform**.

Ex: Transform the parallelogram with vertices $(1,1)$, $(3,-1)$, $(6,0)$, and $(4,2)$ into a square (via a linear transformation).

Sol: $\vec{a} = \langle 4-1, 2-1 \rangle = \langle 3,1 \rangle$

$\vec{b} = \langle 3-1, -1-1 \rangle = \langle 2,-2 \rangle$

PICTURE



$$T(u,v) = u\vec{b} + v\vec{a} = u\langle 2,-2\rangle + v\langle 3,1\rangle$$
$$= \langle 2u+3v, -2u+v \rangle = \langle x,y \rangle$$

$$T: \begin{cases} x = 2u+3v \\ y = -2u+v \end{cases}$$

Addendum: Computing $R_{u,v}$:

$\vec{j}$ in uv $\rightsquigarrow$ $\vec{a}$ in xy
$\vec{i}$ in uv $\rightsquigarrow$ $\vec{b}$ in xy

$$\begin{cases} 1 = 2u+3v \\ 1 = -2u+v \end{cases} \iff \begin{cases} 1 = 2u+3v \\ 2 = 4v \end{cases} \iff \begin{cases} u = \frac{1}{2}(1-3v) \\ v = \frac{1}{2} \end{cases} \iff \begin{cases} u = -\frac{1}{4} \\ v = \frac{1}{2} \end{cases}$$

$$\therefore R_{u,v} = \left[-\tfrac{1}{4}, \tfrac{3}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{2}\right] \quad \leftarrow uv\text{-rectangle}$$

Once we have the transformation, our region is made very nice. However, we've warped the region, so our differential has to change! This is the last piece we need to proceed...

Suppose $u_1, u_2, \cdots, u_n$ is a coordinate system for $\mathbb{R}^n$, and assume

$$\begin{cases} x_1 = x_1(u_1, u_2, \cdots, u_n) \\ x_2 = x_2(u_1, u_2, \cdots, u_n) \\ \vdots \\ x_n = x_n(u_1, u_2, \cdots, u_n) \end{cases}$$

is another coordinate system for $\mathbb{R}^n$ with x_i differentiable for all i.

Thus $(u_1, u_2, \cdots, u_n) \mapsto (x_1, x_2, \cdots, x_n)$ is a smooth coordinate transformation.

The Jacobian of this transformation is

← matrix of first order partial derivatives

$$\star \quad \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)} := \det \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix} \quad \star$$

Ex: Compute the Jacobian of the transformation from above.

Sol:

$$T: \begin{cases} x = 2u + 3v \\ y = -2u + v \end{cases}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} = 2 - -6 = 8 \quad \blacksquare$$

Prop: If $(u_1, u_2, \cdots, u_n) \mapsto (x_1, x_2, \cdots, x_n)$ is a smooth coordinate transformation, then their differentials are related by the equation

$$\star \quad d(x_1, x_2, \cdots, x_n) = \left| \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)} \right| d(u_1, u_2, \cdots, u_n). \quad \star$$

Point: We can now compute integrals using this multivariate substitution method, making certain integrals much easier. :)

Let's go back and recompute the example above ...

Ex: Compute $\iint_R (x+y)\, dA$ where $R \subseteq \mathbb{R}^2$ is the parallelogram with vertices $(1,1)$, $(3,-1)$, $(6,0)$, and $(4,2)$

Sol: From above, we know the transformation

$$T: \begin{cases} x = 2u + 3v \\ y = -2u + v \end{cases} \quad \text{takes } R_{uv} = \left[-\tfrac{1}{4}, \tfrac{3}{4}\right] \times \left[\tfrac{1}{2}, \tfrac{3}{2}\right]$$

to the parallelogram.

Furthermore, T has Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = 8$. Hence:

$$\iint_R (x+y)\, dA_{x,y} = \iint_{R_{uv}} \big((2u+3v) + (-2u+v)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}$$

$$= \int_{u=-\frac{1}{4}}^{3/4} \int_{v=\frac{1}{2}}^{3/2} 8(4v)\, dv\, du = 32 \int_{u=-\frac{1}{4}}^{3/4} \int_{v=\frac{1}{2}}^{3/2} v\, dv\, du$$

$$= 32 \int_{u=-\frac{1}{4}}^{3/4} \frac{1}{2} \left[v^2\right]_{v=\frac{1}{2}}^{3/2} du = 16 \left(\frac{9}{4} - \frac{1}{4}\right) \int_{u=-\frac{1}{4}}^{3/4} du$$

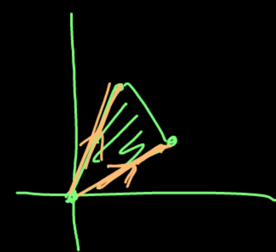
$$= 16 \cdot 2 \, [u]_{u=-\frac{1}{4}}^{3/4} = 32\left(\frac{3}{4} - -\frac{1}{4}\right) = 32 \cdot 1 = 32 \qquad \blacksquare$$

Ex: Compute $\iint_R (x-3y)\,dA$

for R the triangle with vertices $(0,0)$, $(2,1)$, $(1,2)$.

Sol:

Picture





$$T(u,v) = u\langle 2,1\rangle + v\langle 1,2\rangle = \langle x,y\rangle$$

$$T: \begin{cases} x = 2u + v \\ y = v + 2u \end{cases}$$

$$R_{(u,v)} = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1-u\}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$$

$$\therefore \iint_R (x-3y)\,dA_{(x,y)} = \iint_R \left((2u+v) - 3(v+2u)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{(u,v)}$$

$$= \int_{u=0}^{1}\int_{v=0}^{1-u} 3(-u-5v)\,dv\,du$$

$$= -3\int_{u=0}^{1}\left[uv + \frac{5}{2}v^2\right]_{v=0}^{1-u} du$$

$$= -3\int_{u=0}^{1}\left(u(1-u) + \frac{5}{2}(1-u)^2\right) du$$

$$= -3\int_{u=0}^{1}\left(u - u^2 + \frac{5}{2}(u^2 - 2u + 1)\right) du$$

$$= -3\int_{u=0}^{1}\left(\frac{3}{2}u^2 - 4u + \frac{5}{2}\right) du$$

$$= -3\left[\frac{1}{2}u^3 - 2u^2 + \frac{5}{2}u\right]_{u=0}^{1}$$

$$= -3\left(\frac{1}{2} - 2 + \frac{5}{2} - 0\right) = -3(3-2) = -3$$

$\boxed{//}$

Ex: Compute $\iint_R xy\,dA$ for $R = \{(x,y) : 0 \le x,\ x \le y \le 3x,\ 1 \le xy \le 3\}$.

Sol: $T: \begin{cases} xy = u \\ \frac{y}{x} = v \end{cases} \rightsquigarrow T: \begin{cases} x = \sqrt{uv^{-1}} = u^{1/2} v^{-1/2} \\ y = \sqrt{uv} = u^{1/2} v^{1/2} \end{cases}$

$\begin{pmatrix} u \cdot v = y^2 \rightsquigarrow y = \sqrt{uv} \\ u/v = x^2 \rightsquigarrow x = \sqrt{uv^{-1}} \end{pmatrix}$ $x > 0$

$R_{(u,v)} = \{(u,v) : 1 \le v \le 3,\ 1 \le u \le 3\}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2} u^{-1/2} v^{-1/2} & -\frac{1}{2} u^{1/2} v^{-3/2} \\ \frac{1}{2} u^{-1/2} v^{1/2} & \frac{1}{2} u^{1/2} v^{-1/2} \end{bmatrix}$$

$$= \frac{1}{2} u^{-1/2} v^{-1/2} \cdot \frac{1}{2} u^{1/2} v^{-1/2} - \frac{1}{2} u^{-1/2} v^{1/2} \cdot -\frac{1}{2} u^{1/2} v^{-3/2}$$

$$= \frac{1}{4} v^{-1} + \frac{1}{4} v^{-1} = \frac{1}{2} v^{-1}$$

$$\therefore \iint_{R_{(x,y)}} xy\, dA_{(x,y)} = \iint_{R_{(u,v)}} u \cdot \frac{1}{2} v^{-1}\, dA_{(u,v)}$$

$$= \frac{1}{2} \int_{u=1}^{3} \int_{v=1}^{3} uv^{-1}\, dv\, du = \frac{1}{2} \int_{u=1}^{3} u \left[\ln(v)\right]_{v=1}^{3} du$$

$$= \frac{1}{2}\left(\ln(3) - \ln(1)\right) \int_{u=1}^{3} u\, du = \frac{\ln(3)}{2} \cdot \frac{1}{2}\left[u^2\right]_{u=1}^{3}$$

$$= \frac{\ln(3)}{4} \cdot (3^2 - 1^2) = 2\ln(3).$$ ☒

Ex: Compute $\iint_R \cos\left(\frac{y-x}{x+y}\right) dA$

for R the trapezoid with vertices $(1,0)$, $(2,0)$, $(0,2)$, $(0,1)$.

Sol: Left to you as an exercise. ☒

Ex: Compute $\iint_R e^{x+y}\, dA$ for $R = \{(x,y) : |x| + |y| \le 1\}$.

Sol: Left to you as an exercise. ☒

Polar Coordinates

Recall that the polar coordinate transformation is

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

Hence the Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$$= \cos(\theta)\cdot r\cos(\theta) - \sin(\theta)\cdot -r\sin(\theta)$$

$$= r\left(\cos^2(\theta) + \sin^2(\theta)\right) = r$$

Hence the polar transformation satisfies $dA_{(x,y)} = r\, dA_{(r,\theta)}$.

Ex: Compute $\iint_R (x+y)\, dA$ where $R = \{(x,y) : x^2+y^2 \leq 9\}$.

Sol: $\iint_R (x+y)\, dA_{(x,y)} = \iint_R \left(r\cos(\theta) + r\sin(\theta)\right) r\, dA_{(r,\theta)}$

$\left(R_{(r,\theta)} = \{(r,\theta) : 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}\right)$

$$= \int_{r=0}^{3} \int_{\theta=0}^{2\pi} r^2\left(\cos(\theta) + \sin(\theta)\right) d\theta\, dr$$

$$= \int_{r=0}^{3} r^2 \left[\sin(\theta) - \cos(\theta)\right]_{\theta=0}^{2\pi} dr$$

$$= \left(\left(\sin(2\pi) - \cos(2\pi)\right) - \left(\sin(0) - \cos(0)\right)\right) \int_{r=0}^{3} r^2\, dr = 0 \quad \blacksquare$$

Ex: Compute $\iint_D e^{-x^2-y^2}\,dA$ for D the disk of radius 5 about the origin.

Sol: $D = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le 5\}$

$$\iint_D e^{-x^2-y^2}\,dA = \iint_D e^{-r^2}\, r\, dA_{(r,\theta)}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{5} r e^{-r^2}\, dr\, d\theta$$

$$\begin{pmatrix} u = -r^2 & du = -2r\,dr \\ u(0) = 0 & u(5) = -5^2 \end{pmatrix}$$

$$= \int_{\theta=0}^{2\pi} -\frac{1}{2} \int_{r=0}^{5} -2r e^{-r^2}\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{1}{2} \int_{u=0}^{-5^2} e^{u}\, du\, d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{2\pi} \left[e^{u} \right]_{u=0}^{-5^2} d\theta$$

$$= -\frac{1}{2}\left(e^{-5^2} - e^{0}\right) \int_{\theta=0}^{2\pi} d\theta$$

$$= -\frac{1}{2}\left(e^{-5^2} - 1\right) \left[\theta\right]_{\theta=0}^{2\pi}$$

$$= -\frac{1}{2}\left(e^{-5^2} - 1\right)(2\pi - 0)$$

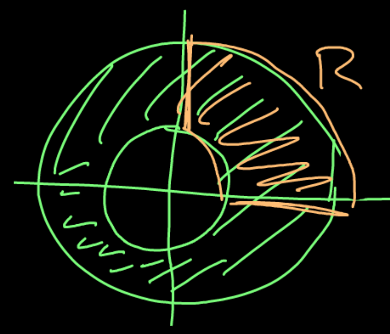
$$= \pi\left(1 - e^{-5^2}\right)$$

//

Ex: Compute $\iint_R \sin(x^2+y^2)\,dA$ for R the region in the first quadrant between the circles of radius 1 and 3 about the origin.

Sol:

$$R = \{(r,\theta) : 0 \le \theta \le \tfrac{\pi}{2},\ 1 \le r \le 3\}$$



$$\therefore \iint_R \sin(x^2+y^2)\,dA$$

$$= \iint_R \sin(r^2)\, r\, dA_{r,\theta}$$

$$= \int_{r=1}^{3} \int_{\theta=0}^{\pi/2} r\sin(r^2)\,d\theta\,dr$$

$$= \int_{r=1}^{3} r\sin(r^2) \int_{\theta=0}^{\pi/2} d\theta\ dr$$

$$= \int_{r=1}^{3} r\sin(r^2)\,[\theta]_{\theta=0}^{\pi/2}\,dr$$

$$= \frac{1}{2}\int_{r=1}^{3} 2r\sin(r^2)\left(\frac{\pi}{2}-0\right)dr$$

$$\begin{pmatrix} u = r^2 & du = 2r\,dr \\ u(1) = 1 & u(3) = 9 \end{pmatrix}$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\int_{u=1}^{9} \sin(u)\,du$$

$$= \frac{\pi}{4}\left[-\cos(u)\right]_{u=1}^{9} = \frac{\pi}{4}\left(\cos(1) - \cos(9)\right)$$

☐

Ex: Compute $\iint_R \cos\left(\sqrt{x^2+y^2}\right) dA$

for R the disk of radius 2 about the origin.

Sol: $R = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$r = \sqrt{x^2+y^2} = \sqrt{r^2}$

$$\therefore \iint_R \cos\left(\sqrt{x^2+y^2}\right) dA_{x,y}$$

$$= \iint_R \cos(r)\, r\, dA_{r,\theta}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r\cos(r)\, dr\, d\theta$$

Compute $\int r\cos(r)\,dr$ using Integration by Parts:

$\hookrightarrow$ $u = r$, $du = dr$; $dv = \cos(r)\,dr$, $v = \sin(r)$

$$= r\sin(r) - \int \sin(r)\,dr = r\sin(r) + \cos(r) + C$$

$$= \int_{\theta=0}^{2\pi} \Big[r\sin(r) + \cos(r) \Big]_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \Big(\big(2\sin(2) + \cos(2)\big) - \big(0\sin(0) + \cos(0)\big) \Big)$$

$$= \Big[\theta\Big]_{\theta=0}^{2\pi} \big(2\sin(2) + \cos(2) - 1\big)$$

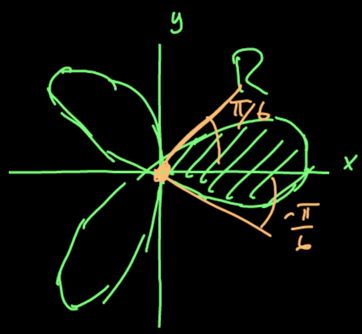
$$= 2\pi\big(2\sin(2) + \cos(2) - 1\big)$$

Ex: Compute the area of one petal of the rose $r = \cos(3\theta)$.

Sol: $\cos(3\theta) = 0 \overset{x\geq 0}{\iff} 3\theta = -\frac{\pi}{2}$ OR $\frac{\pi}{2}$

$\iff \theta = -\frac{\pi}{6}$ OR $\frac{\pi}{6}$

Picture



$\therefore R = \{(r,\theta) : -\pi/6 \leq \theta \leq \frac{\pi}{6},\ 0 \leq r \leq \cos(3\theta)\}$

$$A = \iint_R 1\, dA_{xy} = \iint_R r\, dA_{r,\theta}$$

$$= \int_{\theta=-\frac{\pi}{6}}^{\pi/6} \int_{r=0}^{\cos(3\theta)} r\, dr\, d\theta$$

$$= \int_{\theta=-\frac{\pi}{6}}^{\pi/6} \frac{1}{2}\left[r^2\right]_{r=0}^{\cos(3\theta)} d\theta$$

$\cos^2(z) = \frac{1}{2}(1+\cos(2z))$

$$= \frac{1}{2}\int_{\theta=-\frac{\pi}{6}}^{\pi/6} \cos^2(3\theta)\, d\theta$$

$$= \frac{1}{4}\int_{\theta=-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos(6\theta)+1)\, d\theta$$

$$= \frac{1}{4}\left[\frac{1}{6}\sin(6\theta) + \theta\right]_{\theta=-\frac{\pi}{6}}^{\pi/6}$$

$$= \frac{1}{4}\left(\left(\frac{1}{6}\sin(\pi) + \frac{\pi}{6}\right) - \left(\frac{1}{6}\sin(-\pi) - \frac{\pi}{6}\right)\right)$$

(with $\sin(\pi) \to 0$ and $\sin(-\pi) \to 0$)

$$= \frac{1}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \frac{\pi}{12}$$

∎

Ex: Compute the area of the general ellipse $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} \leq 1$.

Sol:

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

assume $a, b > 0$.

$$\Longleftrightarrow \left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 = 1$$

Make coordinate Transformation: $\begin{cases} \frac{x-x_0}{a} = r\cos(\theta) \\ \frac{y-y_0}{b} = r\sin(\theta) \end{cases}$

Under this transformation: $r^2 = (r\cos(\theta))^2 + (r\sin(\theta))^2 \leq 1 \Leftarrow$

$$\begin{cases} x = ar\cos(\theta) + x_0 \\ y = b\,r\sin(\theta) + y_0 \end{cases}$$

$\leftarrow$ shifted + scaled version of polar coords

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} a\cos(\theta) & -ra\sin(\theta) \\ b\sin(\theta) & rb\cos(\theta) \end{bmatrix}$$

$$= abr\cos^2(\theta) - -abr\sin^2(\theta) = abr\left(\cos^2(\theta) + \sin^2(\theta)\right)$$

$$= abr$$

NB: $R = \left\{ (r,\theta) : \begin{array}{l} 0 \leq r \leq 1, \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$

$$A = \iint_R 1 \, dA_{x,y} = \iint_R \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dA_{r,\theta}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} abr \, dr \, d\theta$$

$$= ab \int_{\theta=0}^{2\pi} \left[\frac{1}{2} r^2\right]_{r=0}^{1} d\theta$$

$$= \frac{ab}{2} \int_{\theta=0}^{2\pi} 1 \, d\theta$$

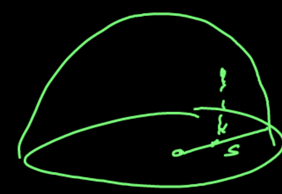
$$= \frac{ab}{2} \left[\theta\right]_{\theta=0}^{2\pi} = \frac{ab}{2}(2\pi - 0) = ab\pi$$

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Ex: Compute the volume of the sphere of radius s. $\left(\underline{\text{Ans}}: \frac{4}{3}\pi s^3\right)$

Sol: $\underbrace{x^2+y^2}+z^2=s^2$

$x^2+y^2=r^2 \quad \therefore f(r,\theta)=\sqrt{s^2-r^2}$



NB: V(Upper hemisphere) = V(lower hemisphere)

$R=\{(r,\theta): 0\le r\le s, \ 0\le\theta\le 2\pi\}$

$$V(\text{sphere}) = 2\iint_R \sqrt{s^2-x^2-y^2}\, dA_{x,y}$$

$$= 2\iint_R \sqrt{s^2-r^2}\; r\, dA_{r,\theta}$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{s} r\sqrt{s^2-r^2}\, dr\, d\theta$$

$$\left(\begin{array}{ll} u=s^2-r^2 & du=-2r\,dr \\ u(0)=s^2 & u(s)=0 \end{array}\right.$$

$$= 2\int_{\theta=0}^{2\pi} -\frac{1}{2}\int_{r=0}^{s}\sqrt{s^2-r^2}\,(-2r)\,dr\, d\theta$$

$$= -\int_{\theta=0}^{2\pi}\int_{u=s^2}^{0} u^{1/2}\, du\, d\theta$$

$$= -\int_{\theta=0}^{2\pi}\left[\frac{2}{3}u^{3/2}\right]_{u=s^2}^{0} d\theta$$

$$= -\frac{2}{3}\left(0-(s^2)^{3/2}\right)\int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{2}{3}s^3\,[\theta]_{\theta=0}^{2\pi}$$

$$= \frac{2}{3}s^3(2\pi-0) = \frac{4}{3}\pi s^3$$

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